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A FORMAL ANALYSIS OF THE EVOLUTION OF COOPERATION(U)
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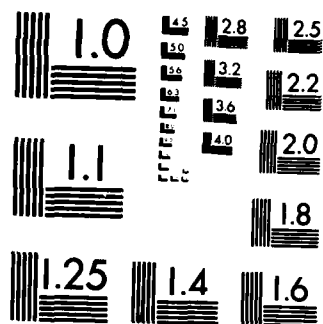
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Jonathan A. K. Cave

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I. INTRODUCTION

In this paper, we examine a formal model of the way a growing population selects behavior. The members of the population engage in randomly-selected binary interactions, with payoffs representing a Prisoners' Dilemma. We assume that players can recognize each other if they have met before, and adopt one of two dynamic strategies; they either play the non-cooperative strategy (G) at every move, which we denote D, or they play a tit-for-tat strategy, in which they begin by playing the cooperative move (C) and continue by playing C if the opponent's last move was C and G if the opponent's last move was G. This latter strategy is denoted T. The interactions are repeated, and two players meet again with probability $\delta \in (0,1)$. Formally, it is as if each player met a random member of the population and played a discounted supergame with discount factor δ forever afterwards. An alternative interpretation is that players repeatedly meet random opponents, but respond to each opponent they have met before on the basis of what happened during their previous encounter.

We assume that the population and the mix of behaviors changes over time in a simple fashion. If one of the two strategies D or T has a higher expected present discounted value (or "fitness") than the other, the proportion of players using that strategy will tend to increase and the proportion using the less fit strategy will tend to diminish. The same conflict mediates average as well as relative fitness. The relative proportions of D and T in the population determine the average expected payoff, $\bar{\phi}$, and the rate of growth of the total population is assumed to be a monotone increasing function of $\bar{\phi}$. Moreover, there exists a critical value $\bar{\phi}^*$ such that a population with $\bar{\phi} > \bar{\phi}^*$ will expand, while a population with average fitness $\bar{\phi} < \bar{\phi}^*$ will contract.

These qualitative dynamics will allow us to say something about the stability of various regimes of behavior in terms of population size and behavior.

The organization of the paper is as follows. Section II contains the model and definitions, and examines the evolution of cooperative behavior for the special case of a static population. Section III combines the dynamics of behavior with those of population growth. Finally, Section IV relates the present results to results of repeated play where players are rational.

II. THE MODEL AND DEFINITIONS

Following Axelrod (1984), we assume that single interactions are mediated by a generalized Prisoners' Dilemma:

	+-----+-----+
C	(b,b) (d,a)
	+-----+-----+
G	(a,d) (c,c)
	+-----+-----+

where $a > b > c > d$, and $2c < a+d < 2b$, so the players cannot escape their dilemma by randomizing. The size of the population is denoted n .

In this model, players are paired randomly, and the result of these matches determines the fitness of the strategies they select. If each player plays only once, these strategies are simply "moves" C or G. As G is a dominant strategy, a population of players who met only once would eventually all play G.

To make the dynamics interesting, we shall assume that players may meet repeatedly during their lifetimes. This assumption changes the strategies used by the players and the way they evaluate outcomes. At the strategic level, players are allowed to make moves that depend on previous history. In this note, we shall limit ourselves to two very simple forms of historical dependence. However, we shall demonstrate in Section V that the implications for the viability of cooperation are not grossly inaccurate. At the payoff level, the possibility of repeated encounters means that the players consider the entire stream of payoffs resulting from their behavior, which determines their ultimate fitness. We shall suppose that players discount the future according to a discount factor $\delta \in (0,1)$. There are various interpretations of δ . In one view, δ represents the (constant) probability that players meeting today will meet again in the future. Alternatively, if we imagine that fitness of an organism during its lifetime is determined by cumulative payoff, δ

may be interpreted as the probability that a player who has not reproduced will reproduce on a given date. In our simplified evolutionary view of the world, the moment of reproduction is the moment of death, since an organism's importance to the species ends when it passes on its behavior.

The two strategies we shall consider here are: a noncooperative strategy D calling for the player to play G every time it is called upon to play, no matter who its opponent may be; and a Tit-for-Tat strategy T calling for an organism to begin by playing C, and to continue by playing C whenever the other player played C on the previous move and G otherwise.¹

For concreteness, we shall describe a population in terms of the number, g , of its members using strategy D.

If two D players meet, the present discounted value each receives is $h(D,D) = c/(1-\delta)$. If two T players meet, the present discounted value each receives is $h(T,T) = b/(1-\delta)$. Finally, if a D player meets a T player, their respective present discounted values are:

$$\begin{aligned} h(D,T) &= [(1-\delta)a + \delta c]/(1-\delta) \\ (1) \quad h(T,D) &= [(1-\delta)d + \delta c]/(1-\delta) \end{aligned}$$

since the Tit-for-Tat player will discover that his opponent is a D player after one move. On the assumption that opponents are randomly selected, we may define the fitness of a strategy to be its expected present discounted value, given n and g :

$$\begin{aligned} F(D;n,g) &= \text{prob}(D) \times h(D,D) + \text{prob}(T) \times h(D,T) \\ &= [(g-1)c + (n-g)(1-\delta)a + (n-g)\delta c]/[(n-1)(1-\delta)] \\ (2) \quad F(T;n,g) &= \text{prob}(D) \times h(T,D) + \text{prob}(T) \times h(T,T) \\ &= [g(1-\delta)d + g\delta c + (n-g-1)b]/[(n-1)(1-\delta)] \end{aligned}$$

The average fitness of the population is:

$$(3) \quad \bar{\phi}(n,g) \equiv [gF(D;n,g) + (n-g)F(T;n,g)]/n$$

Definition: a population (n,g) is in *evolutionary equilibrium* with respect to a critical fitness ϕ^* iff one of the following conditions holds:

- 1 $n = 0$
- 2 $\bar{\phi}(n,g) = \phi^*$, $g = n$, and $F(D;n,g) \geq F(T;n,g)$
- 3 $\bar{\phi}(n,g) = \phi^*$, $g = 0$, and $F(T;n,g) \geq F(D;n,g)$
- 4 $\bar{\phi}(n,g) = \phi^*$, $g \in (0,n)$, and $F(T;n,g) = F(D;n,g)$

These are all stationary equilibria. The first represents extinction of the species. The second represents a stable population that is completely non-cooperative. The third represents a stable population which is completely cooperative, and the fourth represents a stable population in which a mixture of cooperative and noncooperative behaviors survives. In addition to these outcomes, there are equilibria involving continued population growth. Along these "dynamic equilibrium" paths, cooperation may either increase or decrease.

For the balance of this section, we shall assume that the total population is fixed, so we will not need to use $\bar{\phi}(n,g)$.

The dynamics of behavior are determined by relative fitness. That is, we assume that g increases over time iff $F(D;n,g) > F(T;n,g)$. The set of populations for which g is constant is determined by the condition $F(D;n,g) = F(T;n,g)$ in case $g \in (0,n)$. Alternatively, g is constant if $g = n$ and $F(D;n,n-1) > F(T;n,n-1)$ or if $g = 0$ and $F(D;n,1) < F(T;n,1)$. Strictly speaking, this gives conditions for a single mutant to die out, and the definitions do not require that such a mutant never appear. In our model there is no possibility of spontaneous mutation, so these conditions are sufficient. In any event, $g = n$ is always constant, and the analysis of $F(D;n,g) = F(T;n,g)$ which we shall present gives a more precise condition for $g = 0$ to be constant.

There are many ways to present the condition of equal fitness. For the present, we shall give the condition in terms of δ and g , since n is fixed in this section. Let us define $\delta^*(g)$ by:

$$(4) \quad \delta^*(g) = [(n-g)(a-b) + g(c-d) + (b-c)] / [(n-g)(a-c) + g(c-d)]$$

Clearly, $\delta^*(0) = [n(a-b) + (b-c)] / [n(a-c)] < 1$ since $b > c$. Moreover, if we write $\delta^*(g) = f(g)/h(g)$, we note that $0 < f(g) \leq h(g)$ for $g < n$ and also that $f'(g) > h'(g)$. Therefore, $\delta^*(g)$ is a monotone increasing function of g . Indeed, $\delta^*(n-1) = 1$, so the relative fitness of the two strategies is as shown in Figure 1.

Thus, populations starting above the indicated line will tend to become completely cooperative, while populations starting below the line will become completely noncooperative. This diagram also tells us that full cooperation can be sustained if $\delta \geq \delta^*(0)$. It can be sustained in the face of spontaneous mutation of a given degree if δ exceeds $\delta^*(0)$ by a sufficient amount. As we shall show in Section V, this condition is generally stricter than the condition that cooperation can be sustained as an equilibrium with arbitrary strategies: $\delta \geq (a-b)/(a-c)$. However,

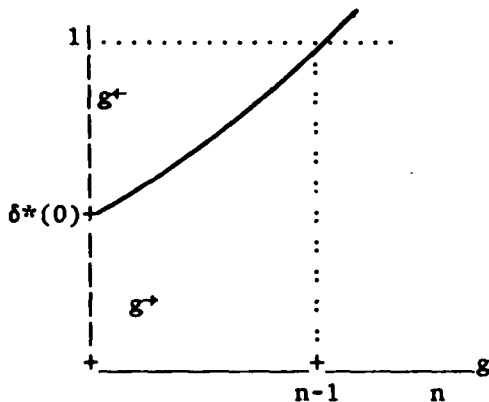


Fig. 1 -- Behavior Dynamics with Fixed Population

as $n \rightarrow \infty$, these conditions become identical. One might therefore say that cooperation is sustainable as an equilibrium with arbitrary strategies iff it is evolutionarily stable (in this sense) in an infinite population.

We can also examine the influence of population size on the stability of cooperation. Equation (4) can be written:

$$(5) \quad g = g^*(n) \equiv [\sigma n - (b-c)]/\psi, \text{ where}$$

$$\sigma = b - (1-\delta)a - \delta c, \text{ and}$$

$$\psi = b + c - a - d + \delta(a + d - 2c)$$

We always have $\sigma < \psi$. Moreover, $\sigma > 0$ iff $\delta > \delta_\sigma$, and $\psi > 0$ iff $\delta > \delta_\psi$, where:

$$(6) \quad \delta_\psi \equiv [a+d-b-c]/[a+d-2c] < [a-b]/[a-c] \equiv \delta_\sigma$$

with the inequality following from $c > d$. There are thus three possibilities:

- $\delta < \delta_\psi$ -- in which case g^* has a positive g -intercept and slope > 1 ;
- $\delta_\psi < \delta < \delta_\sigma$ -- in which case g^* has a negative g -intercept and slope < 1 , and
- $\delta_\sigma < \delta$ -- in which case g^* has a negative g -intercept and positive slope < 1 .

To interpret these in terms of relative fitness, we observe that g will increase iff $\psi g \geq \sigma n - (b-c)$. Therefore:

- if $\delta < \delta_\psi$, g increases iff $g < g^*(n)$; and
- if $\delta > \delta_\psi$, g increases iff $g > g^*(n)$

Since the feasible set of populations is $\{(n, g) : 0 \leq g \leq n\}$, this means that g is increasing everywhere if $\delta \leq \delta_\sigma$. For $\delta > \delta_\sigma$, the dynamics of g with fixed n are as shown in Figure 2.

Therefore, with a fixed population size increasing myopia and small numbers both act to make cooperation less viable.

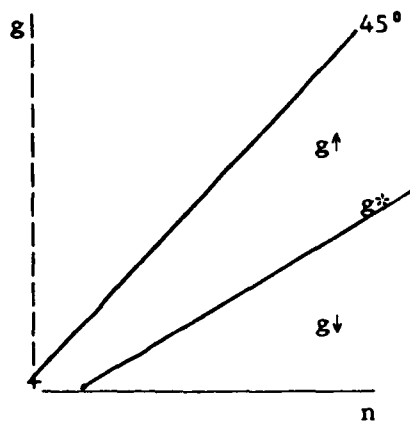


Fig. 2 -- Behavioral Dynamics with Fixed n and $\delta > \delta_\sigma$

III. POPULATION GROWTH AND BEHAVIORAL DYNAMICS

In this section, we shall combine the previous analysis of behavior selection with an analysis of population growth. In doing so, we assume that the rate of growth of population is a strictly monotone increasing function of average fitness $\bar{\phi}(n, g)$ defined in (3) above. This is certainly not the most general assumption; indeed, one can think of many forms of ritual combat for which the effects on individual fitness and species fitness are opposed. However, this assumption serves to simplify the analysis.

Since we are using qualitative dynamics, the condition for nondecreasing population size is that average fitness exceed a critical level $\bar{\phi}^* \leq b$:

$$(7) \quad \bar{\phi}(n, g) - \bar{\phi}^* = \psi g^2 + \omega g + \tau(\bar{\phi}^*) \geq 0, \text{ where}$$

$$\omega = (1-\delta)n(a+d) - (2n-1)b + (2n\delta-1)c$$

$$\tau(\bar{\phi}^*) = n(n-1)(b - \bar{\phi}^*)$$

Setting this expression equal to 0, we obtain two possible values of g at which n is constant:

$$(8) \quad \begin{aligned} h^+(n) &= [-\omega + (\omega^2 - 4\psi\tau(\bar{\phi}^*))^{1/2}]/2\psi, \text{ and} \\ h^-(n) &= [-\omega - (\omega^2 - 4\psi\tau(\bar{\phi}^*))^{1/2}]/2\psi \end{aligned}$$

Let $g^+(n) \equiv \max\{h^+(n), h^-(n)\}$ and $g^-(n) \equiv \min\{h^+(n), h^-(n)\}$.

- n will be constant iff $n = 0$, $g = g^+(n)$, or $g = g^-(n)$.
- If $\delta > \delta_p$, n will be increasing for $g < g^-(n)$ and for $g > g^+(n)$, but will be decreasing for $g^-(n) < g < g^+(n)$.

- If $\delta < \delta_p$, n will be decreasing for $g < g^-(n)$ and for $g > g^+(n)$, but will be increasing for $g^-(n) < g < g^+(n)$.

Proposition: i) $\delta \geq \delta_p$ implies $h^+(n) \geq n$;

ii) $\delta \leq \delta_p$ implies $h^+(n) \leq 0$

iii) $h^-(n) \geq 0$ if

- 1 $\delta \leq \delta_p$; or
- 2 $\delta \geq \delta_p$ and $n \geq 1$;

iv) $h^-(n) \leq n$ if $n \geq 1$.

proof: i) $\delta \geq \delta_p$ implies $\psi \geq 0$. The condition $h^+(n) \geq n$ is then equivalent to:

$$[\omega^2 - 4\psi\tau(\Phi^*)]^{1/2} \geq \omega + 2\psi\tau(\Phi^*), \text{ or } n \geq [\Phi^* - c]/[\Phi^* + (1-2\delta)c]$$

which is always true for $\delta \in [0,1]$.

ii) $\delta \leq \delta_p$ implies $\psi \leq 0$. The condition $h^+(n) \leq 0$ is then equivalent to:

$$[\omega^2 - 4\psi\tau(\Phi^*)]^{1/2} \geq \omega$$

which is certainly true if $\omega < 0$. On the other hand, if $\omega \geq 0$, the above equation can be written:

$$0 \geq 4\psi\tau(\Phi^*)$$

which is always true.

iii) $\delta \geq \delta_p$ implies $\psi \geq 0$, so the condition $h^-(n) \geq 0$ becomes:

$$-\omega \geq [\omega^2 - 4\psi\tau(\Phi^*)]^{1/2}$$

which is true if $\omega \leq 0$. However, $\omega \leq 0$ iff $n \geq N_0$, where

$$N_0 = [b-c]/[2b - (a+d) + \delta(a+d-2c)]$$

Moreover, $N_0 \leq 1$ iff $\psi \geq 0$. Hence $\delta \geq \delta_p$ implies $\psi \geq 0$, which implies $N_0 \geq 1$. Therefore, $n \geq 1$ implies $n \geq N_0$, hence $\omega \leq 0$, and therefore $h^-(n) \geq 0$.

On the other hand, if $\delta < \delta_p$, $\psi < 0$, and the condition $h^-(n) \geq 0$ is:

$$-\omega < [\omega^2 - 4\psi\tau(\phi^*)]^{1/2}$$

which is certainly true if $\omega \geq 0$. Indeed, if $\omega < 0$, we can square both sides of the above condition to give $4\psi\tau(\phi^*) \leq 0$, which follows from $\delta < \delta_p$.

iv) First, suppose that $\psi \geq 0$; the condition $h^-(n) \leq n$ is:

$$\begin{aligned} & -[\omega^2 - 4\psi\tau(\phi^*)]^{1/2} \leq 2\psi n + \omega, \text{ or} \\ & 0 > \tau(\phi^*) + \psi n^2 + \omega n, \text{ or} \\ & 0 < (\phi^* - c) + (1-\delta)(a+d) \end{aligned}$$

which is clearly true.

If $\psi < 0$, $h^-(n) \leq n$ only if a) $2\psi n + \omega \leq 0$ and

b) $-\omega^2 - 4\psi\tau(\phi^*) \geq 2\psi n + \omega$. a) is true iff

$$n \geq [b - c]/[(1-\delta)(a + d - 2c)]$$

which is < 1 iff $\psi < 0$. Therefore, $n \geq 1$ implies a), while b) is:

$$0 \leq (1-\delta)(a+d) + (\phi^* - c)$$

which is true by assumption.

QED

There are various types of candidate equilibria for this process. There are internal equilibria at which $g = g^-(n) = g^*(n)$, which have constant populations with a mixture of behaviors. There are also extinction equilibria in which n approaches 0, and expansion equilibria in which n tends to infinity. The behavior associated with each type of equilibrium path will depend on the height of the critical fitness among other things.

Before illustrating these possibilities, we shall discuss the possibility of cyclical behavior. An equilibrium in which $g \in (0, n)$ represents the intersection of $g^*(n)$ with $g^-(n)$. From equation (11), we infer that g^* passes through the feasible range only if $\delta > \delta_\sigma > \delta_p$.

The derivative of g^- w.r.t. n is nonnegative iff:

$$(9) \quad 4\psi\tau_n(\phi^*) \geq \omega_n[2 + (\omega_n^2 + 4\psi\tau(\phi^*))^{1/2}], \text{ where}$$

$$\begin{aligned} \omega_n &= (1-\delta)(a+d) - 2b + 2\delta c, \text{ and} \\ \tau_n(\phi^*) &= (2n-1)(b-\phi^*) \end{aligned}$$

which follows from the facts that $a+d < 2b$ and $\phi^* \leq b$. In fact, this formula can also be used to show that $g^-(n)$ is strictly concave. The phase diagrams corresponding to possible intersections of g^* and g^- are shown in Figure 3. Both g^* and g^- are increasing. If g^* crosses g^- from above, the equilibrium at e is only approached along the separatrix S , and all other paths ultimately diverge monotonically from e . If g^* crosses g^- from below, e is completely unstable, and all paths diverge in an ultimately monotone fashion.

On the other hand, by continuity any cyclical behavior must occur around an equilibrium of $g^*(n)$ and $g^-(n)$, so there can be no cycles.

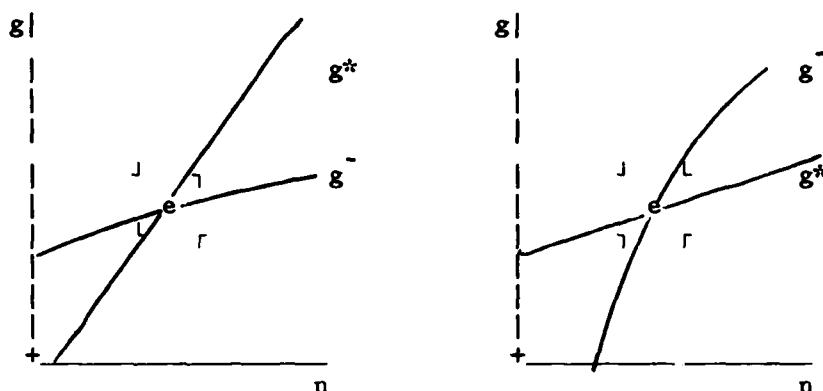


Fig. 3 -- Qualitative Dynamics near an Interior Equilibrium of g^* and g_-

We shall now illustrate the possible equilibrium behavior in the context of a specific example.

THE EXAMPLE

This example is defined by the following data:

$$a = 4, b = 3, c = 1, d = 0$$

Under these conditions, the critical values of δ are

$$\delta_p = (a+d-b-c)/(a+d-2c) = 0, \text{ and } \delta_o = (a-b)/(a-c) = 1/3$$

The functions defining the behavior of g and n are:

$$g^*(n) = (3\delta-1)n/2\delta - 1/\delta$$

$$h^+(n) = [n(1+\delta)-1]/2\delta + [(n(1+\delta)-1)^2 - 2\delta n(n-1)(3-\delta)]^{1/2}/2\delta$$

$$h^-(n) = [n(1+\delta)-1]/2\delta - [(n(1+\delta)-1)^2 - 2\delta n(n-1)(3-\delta)]^{1/2}/2\delta$$

We shall examine three versions of this model, corresponding to three possible values for Φ^* .

Case 1: $\Phi^* = 3$

In this case, full cooperation is essential for survival. This is reflected in the behavioral equations:

$$g^+(n) = \max\{n + (n-1)/\delta, 0\}; \quad g^-(n) = \min\{n + (n-1)/\delta, 0\}$$

The phase diagram is shown in Figure 4. Populations starting above the separatrix labelled "S" are doomed to extinction, since they cross the g^* locus before reaching full cooperation. Once they reach $g^*(n)$, cooperation begins to erode, and the species must decline. Populations starting below S will survive as totally cooperative ($g = 0$) populations. Their size is determined by the point at which they hit the n -axis.

The size and position of the region of viable populations is determined by the discount factor δ (as well as by the unspecified relative rates of adjustment of g and n). As δ grows, the set of viable populations expands, since g^* becomes steeper, approaching $n-(1/2)$. If the population becomes more myopic, g^* becomes flatter and intersects

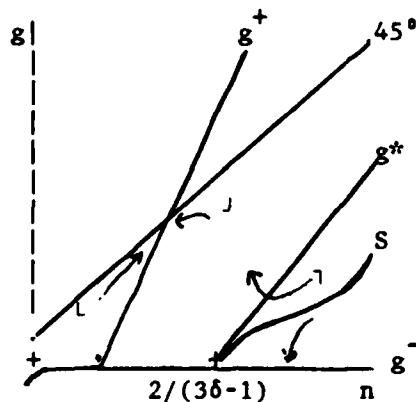


Fig. 4 -- Population Dynamics for Case 1: $\Phi^* = 3$, $\delta > 1/3$

the n -axis at a greater total population. Indeed, for $\delta \leq 1/3$, the only outcome is extinction.

It should be mentioned that there is a region near the origin that lies below the 45° line but above $g^+(n)$. This region only includes populations of size less than one. However, populations within this region do enjoy positive fitness, and to be strictly accurate we should say that all populations which do not reach a completely cooperative equilibrium reach a completely noncooperative equilibrium with $n = 1$. However, this may be considered extinction in our model.

Case 2: $\phi^* = 1$

This case represents the other extreme: even completely noncooperative behavior is viable. In this case, the behavioral dynamics are determined by:

$$g^+(n) = \max\{(n-1)/\delta, n\}; \quad g^-(n) = \min\{(n-1)/\delta, n\}$$

The phase diagram is shown in Figure 5. Populations beginning in region A will become extinct, following a noncooperative path. Populations in region B will reach a noncooperative equilibrium where their trajectory touches the 45° line. Populations starting in region C will grow forever, with a noncooperative population whose relative magnitude is bounded below by $(3\delta-1)/2\delta$. Populations in region D will grow monotonically, but the number of noncooperative players will first increase until the trajectory reaches $g^*(n)$ and then decrease monotonically to 0. Finally, populations in region E will grow and become completely cooperative.

Case 3: $\phi^* = 2$

This case represents an intermediate level of "selection," and displays more interesting behavior. The $g^+(n)$ locus lies entirely above the 45° line for $n \geq 1$, and the $g^-(n)$ locus is given by:

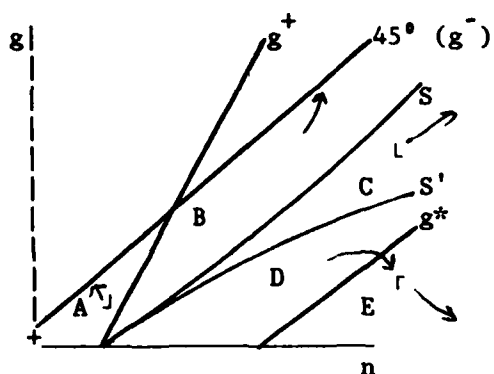


Fig. 5 -- Population Dynamics for Case 2: $\phi^* = 1$, $\delta > 1/3$

$$g^-(n) = [n(1+\delta) - 1 - [(1+\delta^2)n^2 - 2n + 1]^{1/2}]/2\delta$$

This is a concave monotone increasing curve passing through $(n, g) = (1, 0)$, and asymptotic to $[1 + \delta - (1 + \delta^2)^{1/2}]/2\delta$. This curve intersects $g^*(n)$ as long as $\delta > 2 - (3)^{1/2}$, and is labelled "e" in the phase diagram presented in Figure 6.

Populations lying above the separatrix S will become extinct. If they lie above g^* and g^- , the population will shrink monotonically while the proportion of noncooperative players increases. If the population

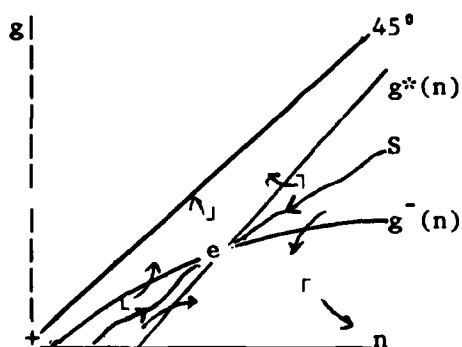


Fig. 6 -- Population Dynamics for Case 3: $\phi^* = 2$, $\delta > 1/3$

lies above S but below g^* , population will shrink monotonically, while the degree of cooperation first increases and then decreases. Finally, populations starting above S but below \bar{g} will have a monotonically increasing proportion of noncooperative players. Total population will increase initially, but then decline to 0.

Populations starting on the separatrix will converge monotonically to the "mixed-behavior" equilibrium at "e". This is an unstable path of the sort familiar to growth economists. Random increases in noncooperation will lead to extinction; random increases in cooperation will lead to continued cooperative growth.

Finally, populations starting below S will eventually expand, and the degree of cooperation will eventually become absolute. However, if such a population starts above g^* there will be an initial period of increasing noncooperation. On the other hand, a population starting above S but below \bar{g} will experience an initial decline in population followed by growth.

IV. RELATION TO NASH EQUILIBRIUM IN THE SUPERGAME

The concept of evolutionary equilibrium used here has certain relations to equilibrium concepts used in the game theoretic literature. These relations can shed light on the validity of the assumption that the only two possible strategies are D and T. In this discussion we assume a fixed population size.

A more general definition of evolutionary dynamics would begin by specifying a set, Σ , of possible strategies to be used in a binary symmetric game with payoff function h . Assuming that each member of the population uses a pure strategy $\sigma \in \Sigma$, a population strategy $s \in S$ is defined, where S is the set of probability measures on Σ . For a finite population, the feasible population strategies will form a finite subset of S .

For a game with a finite population n , $s(\sigma)$ will be a rational number of the form $s(\sigma) = m(\sigma)/n$, where $m(\sigma)$ is the number of individuals using strategy σ . In an infinite game, $s(\sigma)$ is the proportion of the population using σ . Using these interpretations, we can define the *fitness* of a pure strategy σ against s in a game of size n as:

$$F(\sigma, s; n) = [(m(\sigma) - 1)h(\sigma, \sigma) + \sum_{\tau \neq \sigma} m(\tau)h(\sigma, \tau)] / (n - 1) \quad \text{for } n < \infty;$$

$$F(\sigma, s; \infty) = \sum_{\tau} s(\tau)h(\sigma, \tau)$$

An evolutionary dynamical system is an equation $G: S \rightarrow S$ such that $s(\sigma), s(\tau) > 0$, $F(\sigma, s; n) > F(\tau, s; n)$ implies $G(s)(\sigma)s(\tau) > G(s)(\tau)s(\sigma)$. An evolutionary equilibrium is a rest point of an evolutionary dynamical system. This is a very broad definition of an evolutionary equilibrium, that includes all Nash equilibria of the game if the population is

infinite, and all feasible Nash equilibria if not. In fact, in the finite case it includes some outcomes that are not Nash equilibria. Another interesting point is that an evolutionary dynamical system may give rise to cyclical behavior in the neighborhood of an infeasible Nash equilibrium.

Tighter definitions of evolutionary equilibrium can be obtained by adding stability considerations. For example, we might wish to add the condition that the proposed equilibrium behavior should be approachable from a specified set of initial conditions, or that it should be stable under a specified set of perturbations. This gives the evolutionary concept some of the flavor of perfect equilibrium concepts.

For our purposes, it suffices to observe that evolution from a completely mixed starting point may be equivalent to sequential elimination of dominated strategies. If a given pure strategy is dominated by another pure strategy, the number of individuals using the dominated strategy will eventually decline in a completely mixed population following evolutionary dynamics. In fact, it will become extinct before the dominating strategy. Eventually, we can look forward to a situation in which all strategies dominated by pure strategies have been eliminated from the population. At this point, the same argument can be applied repeatedly. However, there is nothing in the qualitative dynamics to ensure that the first strategies to go will all be dominated strategies. Therefore, since order is important in determining the results of sequential elimination of dominated strategies, we cannot use the idea of starting from a completely mixed strategy to conclude that only sequentially undominated strategies will appear, although it does ensure that any evolutionary equilibrium established in this way is perfect.

However, it remains true that there exist evolutionary dynamical systems which have the property that dominated strategies are eliminated before undominated strategies when the population starts from a completely mixed position. Using such a system, one can infer that evolutionary equilibrium will involve sequentially undominated

strategies. In the undiscounted Prisoner's Dilemma in which player's strategies are allowed to depend only on the opponent's previous move, the strategies D and T are the last strategies to be eliminated.²

In that game, T is the only surviving strategy at the last step, but with discounting T does not dominate D. This provides indirect evidence that a population that:

- is limited to remembering the opponent's previous move;
- begins with a complete mix of strategies; and
- follows dynamics which result in relatively early elimination of dominated strategies

will eventually conform to the assumptions of the model used here.

FOOTNOTES

1. In the version of the game where players cannot recognize each other, a Tit-for-Tat player's move is based on the previous move of whichever opponent it faced previously. Of course, in this model there are no continuing relationships, and the players face random opponent selection at each stage.

2. See Cave (1984).

REFERENCES

Axelrod, R., *The Evolution of Cooperation*, New York, Basic Books, 1984.

Cave, J., "Mutual Recognition of Human Fallibility: a resolution of the Prisoners' Dilemma," Rand P-7030, November, 1984.

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